TWO-VECTOR BUNDLES AND FORMS OF ELLIPTIC COHOMOLOGY

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Dedicated to Graeme Segal on the occasion of his 60th birthday

1. Introduction

The work to be presented in this paper has been inspired by several of Professor Graeme Segal's papers. Our search for a geometrically defined elliptic cohomology theory with associated elliptic objects obviously stems from his Bourbaki seminar [Se88]. Our readiness to form group completions of symmetric monoidal categories by passage to algebraic K-theory spectra derives from his Topology paper [Se74]. Our inclination to invoke 2-functors to the 2-category of 2-vector spaces generalizes his model for topological K-theory in terms of functors from a path category to the category of vector spaces. We offer him our admiration.

Among all generalized (co-)homology theories, a few hold a special position because they are, in some sense, geometrically defined. For example, de Rham cohomology of manifolds is defined in terms of cohomology classes of closed differential forms, topological K-theory of finite CW complexes is defined in terms of equivalence classes of complex vector bundles, and complex bordism is defined in terms of bordism classes of maps from stably complex manifolds. The geometric origin of these theories makes them particularly well suited to the analysis of many key problems. For example, Chern–Weil theory associates differential forms related to the curvature tensor to manifolds with a connection, whose de Rham cohomology classes are the Chern classes of the tangent bundle of the manifold. The Atiyah–Segal index theory [AS68] associates formal differences of vector bundles to parametrized families of Fredholm operators, arising e.g. from complexes of elliptic pseudo-differential operators, and their isomorphism classes live in topological K-theory. Moduli spaces of isomorphism classes of solutions to e.g. Yang–Mills gauge-theoretic problems can generically yield maps from suitably structured manifolds, with well-defined bordism classes in the corresponding form of bordism homology. On the other hand, Quillen's theorem that the coefficient ring $\pi_*(MU)$ for complex bordism theory is the Lazard ring that corepresents (commutative 1-dimensional) formal group laws has no direct manifold-geometric interpretation, and may seem to be a fortuitous coincidence in this context.

From the chromatic point of view of stable homotopy theory, related to the various periodicity operators v_n for $n \geq 0$ that act in many cohomology theories, these three geometrically defined cohomology theories detect an increasing amount of information. De Rham cohomology or real singular cohomology sees only rational phenomena, because for each prime p multiplication by $p = v_0$ acts invertibly on

$H^*(X;\mathbb{R})$. Topological K-theory only picks up Bott periodic phenomena, because multiplication by the Bott class $u \in \pi_2(KU)$ acts invertibly on $KU^*(X)$, and $u^{p-1} = v_1$ for each prime p. Complex bordism $MU_*(X)$ instead detects all levels of periodic phenomena. We can say that real cohomology, topological K-theory and complex bordism have *chromatic filtration* 0, 1 and ∞, respectively. A precise interpretation of this is that the spectra $H\mathbb{R}$ and KU are Bousfield local with respect to the Johnson–Wilson spectra $E(n)$ for $n = 0$ and 1, respectively, while MU is not $E(n)$-local for any finite n. Traditionally, an elliptic cohomology theory is a complex oriented Landweber exact cohomology theory associated to the formal group law of an elliptic curve. It will have chromatic filtration 2 when the elliptic curve admits a supersingular specialization, and so any cohomology theory of chromatic filtration 2 might loosely be called a form of elliptic cohomology. However, the formal group law origin of traditional elliptic cohomology is not of a directly geometric nature, and so there has been some lasting interest in finding a truly geometrically defined form of elliptic cohomology.

It is the aim of the present paper to introduce a geometrically defined cohomology theory that is essentially of chromatic filtration 2, or more precisely, a connective form of such a theory. It therefore extends the above list of distinguished cohomology theories one step beyond topological K-theory, to a theory that will detect v_2-periodic phenomena, but will ignore the complexity of all higher v_n-periodicities for $n \geq 3$.

The theory that we will present is represented by the algebraic K-theory spectrum $K(\mathcal{V})$ of the Kapranov–Voevodsky 2-category of 2-vector spaces [KV94]. A 2-vector space is much like a complex vector space, but with all occurrences of complex numbers, sums, products and equalities replaced by finite-dimensional complex vector spaces, direct sums, tensor products and coherent isomorphisms, respectively. It is geometrically defined in the sense that the 0-th cohomology group $K(\mathcal{V})^0(X)$ of a space X can be defined in terms of equivalence classes of *2-vector bundles* over X (or more precisely, over the total space Y of a Serre fibration $Y \to X$ with acyclic homotopy fibers, i.e., an acyclic fibration). Cf. theorem 4.10. A 2-vector bundle over X is a suitable bundle of categories, defined much like a complex vector bundle over X, but subject to the same replacements as above. The previously studied notion of a gerbe over X with band $\mathbb{C}^*$ is a special case of a 2-vector bundle, corresponding in the same way to a complex line bundle.

We conjecture in 5.1 that the spectrum $K(\mathcal{V})$ is equivalent to the algebraic K-theory spectrum $K(ku)$ of the connective topological K-theory spectrum ku, considered as a "brave new ring", i.e., as an $\mathbb{S}$-algebra. This is a special case of a more general conjecture, where for a symmetric bimonoidal category $\mathcal{B}$ (which is a generalization of a commutative semi-ring) we compare the category of finitely generated free modules over $\mathcal{B}$ to the category of finitely generated free modules over the commutative $\mathbb{S}$-algebra $\mathbb{A} = \mathrm{Spt}(\mathcal{B})$ (which is a generalization of a commutative ring) associated to $\mathcal{B}$. The conjecture amounts to a form of "positive thinking", asserting that for the purpose of forming algebraic K-theory spectra it should not matter whether we start with a semi-ring-like object (such as the symmetric bimonoidal category $\mathcal{B}$) or the ring-like object given by its additive Grothendieck group completion (such as the commutative $\mathbb{S}$-algebra $\mathbb{A}$). This idea originated with Marcel Bökstedt, and we are indebted to him for suggesting this approach. We have verified the conjecture in the case of actual commutative semi-rings, in-

terpreted as symmetric bimonoidal categories that only have identity morphisms, and view this as strong support in favor of the conjecture.

Continuing, we know that $K(ku)$, or rather a spectrum very closely related to it, is essentially of chromatic filtration 2. For connective spectra, such as all those arising from algebraic K-theory, there is a more appropriate and flexible variation of the chromatic filtration that we call the *telescopic complexity* of the spectrum; cf. definition 6.1. For example, integral and real cohomology have telescopic complexity 0, connective and periodic topological K-theory have telescopic complexity 1, and traditional elliptic cohomology has telescopic complexity 2.

It is known, by direct nontrivial calculations [AR02], that $K(\ell_p^\wedge)$ has telescopic complexity 2, where $\ell_p^\wedge$ is the connective p-complete Adams summand of topological K-theory and $p \geq 5$. The use of the Adams summand in place of the full connective p-complete topological K-theory spectrum $ku_p^\wedge$, as well as the hypothesis $p \geq 5$, are mostly technical assumptions that make the calculations manageable, and it seems very likely that also $K(ku_p^\wedge)$ will have telescopic complexity 2 for any prime p. It then follows from [D97], if we assume the highly respectable Lichtenbaum–Quillen conjecture for $K(\mathbb{Z})$ at p, that also $K(ku)$ has telescopic complexity 2. In this sense we shall allow ourselves to think of $K(ku)$, and conjecturally $K(\mathcal{V})$, as a connective form of elliptic cohomology.

The definition of a 2-vector bundle is sufficiently explicit that it may carry independent interest. In particular, it may admit notions of *connective structure* and *curving*, generalizing the notions for gerbes [Br93, §5.3], such that to each 2-vector bundle $\mathcal{E}$ over X with connective structure there is an associated virtual vector bundle H over the free loop space $\mathcal{L}X = \mathrm{Map}(S^1, X)$, generalizing the *anomaly line bundle* for gerbes [Br93, §6.2]. If $\mathcal{E}$ is equipped with a curving, there probably arises an *action functional* for oriented compact surfaces over X (loc. cit.), providing a construction of an *elliptic object* over X in the sense of Segal [Se88]. Thus 2-vector bundles over X (with extra structure) may have naturally associated elliptic objects over X. However, we have not yet developed this theory properly, and shall therefore postpone its discussion to a later joint paper, which will also contain proofs of the results announced in the present paper. Some of the basic ideas presented here were sketched by the first author in [Ba98].

The paper is organized as follows. In §2 we define a charted 2-vector bundle of rank n over a space X with respect to an open cover $\mathcal{U}$ that is indexed by a suitably partially ordered set $\mathcal{I}$. This corresponds to a Steenrod-style definition of a fiber bundle, with standard fiber the category $\mathcal{V}^n$ of n-tuples of finite-dimensional complex vector spaces, chosen trivializations over the chart domains in $\mathcal{U}$, gluing data that compare the trivializations over the intersection of two chart domains and coherence isomorphisms that systematically relate the two possible comparisons that result over the intersection of three chart domains. We also discuss when two such charted 2-vector bundles are to be viewed as equivalent, i.e., when they define the same abstract object.

In §3 we think of a symmetric bimonoidal category $\mathcal{B}$ as a generalized semiring, and make sense of the algebraic K-theory $K(\mathcal{B})$ of its 2-category of finitely generated free "modules" $\mathcal{B}^n$. We define the weak equivalences $\mathcal{B}^n \to \mathcal{B}^n$ to be given by a monoidal category $\mathcal{M} = GL_n(\mathcal{B})$ of *weakly invertible* matrices over $\mathcal{B}$, cf. definition 3.6, in line with analogous constructions for simplicial rings and $\mathbb{S}$-algebras [W78]. It is a key point that we allow $GL_n(\mathcal{B})$ to contain more matrices

than the strictly invertible ones, of which there are too few to yield an interesting theory. We also present an explicit bar construction $B\mathcal{M}$ that is appropriate for such monoidal categories. Our principal example is the symmetric bimonoidal category $\mathcal{V}$ of finite-dimensional complex vector spaces under direct sum and tensor product, for which the modules $\mathcal{V}^n$ are the 2-vector spaces of Kapranov and Voevodsky.

In §4 we bring these two developments together, by showing that the equivalence classes of charted 2-vector bundles of rank n over a (reasonable) space X is in natural bijection (theorem 4.5) with the homotopy classes of maps from X to the geometric realization $|BGL_n(\mathcal{V})|$ of the bar construction on the monoidal category of weakly invertible $n \times n$ matrices over $\mathcal{V}$. The group of homotopy classes of maps from X to the algebraic K-theory space $K(\mathcal{V})$ is naturally isomorphic (theorem 4.10) to the Grothendieck group completion of the abelian monoid of virtual 2-vector bundles over X, i.e., the 2-vector bundles $\mathcal{E} \downarrow Y$ over spaces Y that come equipped with an acyclic fibration $a\colon Y \to X$. Hence the contravariant homotopy functor represented by $K(\mathcal{V})$ is geometrically defined, in the sense that virtual 2-vector bundles over X are the (effective) cycles for this functor at X.

In §5 we compare the algebraic K-theory of the generalized semi-ring $\mathcal{B}$ to the algebraic K-theory of its additive group completion. To make sense of the latter as a ring object, as is necessary to form its algebraic K-theory, we pass to structured ring spectra, i.e., to the commutative $\mathbb{S}$-algebra $\mathbb{A} = \mathrm{Spt}(\mathcal{B})$. We propose that the resulting algebraic K-theory spectra $K(\mathcal{B})$ and $K(\mathbb{A})$ are weakly equivalent (conjecture 5.1), and support this assertion by confirming that it holds true in the special case of a discrete symmetric bimonoidal category $\mathcal{B}$, i.e., a commutative semi-ring in the usual sense. In the special case of 2-vector spaces the conjecture asserts that $K(\mathcal{V})$ is the algebraic K-theory $K(ku)$ of connective topological K-theory ku viewed as a commutative $\mathbb{S}$-algebra.

In §6 we relate the spectrum $K(ku)$ to the algebraic K-theory spectrum $K(\ell_p^\wedge)$ of the connective p-complete Adams summand $\ell_p^\wedge$ of $ku_p^\wedge$. The latter theory $K(\ell_p^\wedge)$ is known (theorem 6.4, [AR02]) to have telescopic complexity 2, and this section makes it plausible that also the former theory $K(ku)$ has telescopic complexity 2, and hence is a connective form of elliptic cohomology. Together with conjecture 5.1 this says that (a) the generalized cohomology theory represented by $K(ku)$ is geometrically defined, because its 0-th cohomology group, which is then represented by $K(\mathcal{V})$, is defined in terms of formal differences of virtual 2-vector bundles, and (b) that it has telescopic complexity 2, meaning that it captures one more layer of chromatic complexity than topological K-theory does.

2. Charted two-vector bundles

Definition 2.1. Let X be a topological space. An *ordered open cover* $(\mathcal{U}, \mathcal{I})$ of X is a collection $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{I}\}$ of open subsets $U_\alpha \subset X$, indexed by a partially ordered set $\mathcal{I}$, such that

(1) the U_α cover X in the sense that $\bigcup_\alpha U_\alpha = X$, and
(2) the partial ordering on $\mathcal{I}$ restricts to a total ordering on each finite subset $\{\alpha_0, \dots, \alpha_p\}$ of $\mathcal{I}$ for which the intersection $U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap \dots \cap U_{\alpha_p}$ is nonempty.

The partial ordering on $\mathcal{I}$ makes the nerve of the open cover $\mathcal{U}$ an ordered simplicial complex, rather than just a simplicial complex. We say that $\mathcal{U}$ is a *good cover* if each finite intersection $U_{\alpha_0 \dots \alpha_p}$ is either empty or contractible.

Definition 2.2. Let X be a topological space, with an ordered open cover $(\mathcal{U}, \mathcal{I})$, and let $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ be a non-negative integer. A *charted 2-vector bundle* $\mathcal{E}$ of rank n over X consists of

(1) an $n \times n$ matrix
$$E^{\alpha\beta} = (E^{\alpha\beta}_{ij})^n_{i,j=1}$$
of complex vector bundles over $U_{\alpha\beta}$, for each pair $\alpha < \beta$ in $\mathcal{I}$, such that over each point $x \in U_{\alpha\beta}$ the integer matrix of fiber dimensions
$$\dim(E^{\alpha\beta}_x) = (\dim E^{\alpha\beta}_{ij,x})^n_{i,j=1}$$
is invertible, i.e., has determinant ± 1, and

(2) an $n \times n$ matrix
$$\phi^{\alpha\beta\gamma} = (\phi^{\alpha\beta\gamma}_{ik})^n_{i,k=1} \colon E^{\alpha\beta} \cdot E^{\beta\gamma} \xrightarrow{\cong} E^{\alpha\gamma}$$
of vector bundle isomorphisms
$$\phi^{\alpha\beta\gamma}_{ik} \colon \bigoplus_{j=1}^{n} E^{\alpha\beta}_{ij} \otimes E^{\beta\gamma}_{jk} \xrightarrow{\cong} E^{\alpha\gamma}_{ik}$$
over $U_{\alpha\beta\gamma}$, for each triple $\alpha < \beta < \gamma$ in $\mathcal{I}$, such that

(3) the diagram
$$\begin{tikzcd}
E^{\alpha\beta} \cdot (E^{\beta\gamma} \cdot E^{\gamma\delta}) \arrow[rr, "\underline{\alpha}"] \arrow[d, "id \cdot \phi^{\beta\gamma\delta}"'] & & (E^{\alpha\beta} \cdot E^{\beta\gamma}) \cdot E^{\gamma\delta} \arrow[d, "\phi^{\alpha\beta\gamma} \cdot id"] \\
E^{\alpha\beta} \cdot E^{\beta\delta} \arrow[r, "\phi^{\alpha\beta\delta}"'] & E^{\alpha\delta} & E^{\alpha\gamma} \cdot E^{\gamma\delta} \arrow[l, "\phi^{\alpha\gamma\delta}"]
\end{tikzcd}$$
of vector bundle isomorphisms over $U_{\alpha\beta\gamma\delta}$ commutes, for each chain $\alpha < \beta < \gamma < \delta$ in $\mathcal{I}$.

Here $\underline{\alpha}$ denotes the (coherent) natural associativity isomorphism for the matrix product $\cdot$ derived from the tensor product $\otimes$ of vector bundles. We call the $n \times n$ matrices $E^{\alpha\beta}$ and $\phi^{\alpha\beta\gamma}$ the *gluing bundles* and the *coherence isomorphisms* of the charted 2-vector bundle $\mathcal{E} \downarrow X$, respectively.

Definition 2.3. Let $\mathcal{E}$ and $\mathcal{F}$ be two charted 2-vector bundles of rank n over X, with respect to the same ordered open cover $(\mathcal{U}, \mathcal{I})$, with gluing bundles $E^{\alpha\beta}$ and $F^{\alpha\beta}$ and coherence isomorphisms $\phi^{\alpha\beta\gamma}$ and $\psi^{\alpha\beta\gamma}$, respectively. An *elementary change of trivializations* $(T^\alpha, \tau^{\alpha\beta})$ from $\mathcal{E}$ to $\mathcal{F}$ is given by

(1) an $n \times n$ matrix $T^\alpha = (T^\alpha_{ij})^n_{i,j=1}$ of complex vector bundles over U_α, for each α in $\mathcal{I}$, such that over each point $x \in U_\alpha$ the integer matrix of fiber dimensions $\dim(T^\alpha_x)$ has determinant ± 1, and

(2) an $n \times n$ matrix of vector bundle isomorphisms
$$\tau^{\alpha\beta} = (\tau^{\alpha\beta}_{ij})^n_{i,j=1} \colon F^{\alpha\beta} \cdot T^\beta \xrightarrow{\cong} T^\alpha \cdot E^{\alpha\beta}$$
over $U_{\alpha\beta}$, for each pair $\alpha < \beta$ in $\mathcal{I}$, such that

(3) the diagram

$$\begin{tikzcd}
F^{\alpha\beta} \cdot F^{\beta\gamma} \cdot T^{\gamma} \arrow[r, "id\cdot\tau^{\beta\gamma}"] \arrow[d, "\psi^{\alpha\beta\gamma}\cdot id"'] & F^{\alpha\beta} \cdot T^{\beta} \cdot E^{\beta\gamma} \arrow[r, "\tau^{\alpha\beta}\cdot id"] & T^{\alpha} \cdot E^{\alpha\beta} \cdot E^{\beta\gamma} \arrow[d, "id\cdot\phi^{\alpha\beta\gamma}"] \\
F^{\alpha\gamma} \cdot T^{\gamma} \arrow[rr, "\tau^{\alpha\gamma}"'] & & T^{\alpha} \cdot E^{\alpha\gamma}
\end{tikzcd}$$

(natural associativity isomorphisms suppressed) of vector bundle isomorphisms over $U_{\alpha\beta\gamma}$ commutes, for each triple $\alpha < \beta < \gamma$ in $\mathcal{I}$.

Definition 2.4. Let $(\mathcal{U}, \mathcal{I})$ and $(\mathcal{U}', \mathcal{I}')$ be two ordered open covers of X. Suppose that there is an order-preserving *carrier function* $c\colon \mathcal{I}' \to \mathcal{I}$ such that for each $\alpha \in \mathcal{I}'$ there is an inclusion $U'_\alpha \subset U_{c(\alpha)}$. Then $(\mathcal{U}', \mathcal{I}')$ is a *refinement* of $(\mathcal{U}, \mathcal{I})$.

Let $\mathcal{E}$ be a charted 2-vector bundle of rank n over X with respect to $(\mathcal{U}, \mathcal{I})$, with gluing bundles $E^{\alpha\beta}$ and coherence isomorphisms $\phi^{\alpha\beta\gamma}$. Let

$$c^*E^{\alpha\beta} = E^{c(\alpha)c(\beta)}|U'_{\alpha\beta}$$

for $\alpha < \beta$ in $\mathcal{I}'$ and

$$c^*\phi^{\alpha\beta\gamma} = \phi^{c(\alpha)c(\beta)c(\gamma)}|U'_{\alpha\beta\gamma}$$

for $\alpha < \beta < \gamma$ in $\mathcal{I}'$, be $n \times n$ matrices of vector bundles and vector bundle isomorphisms over $U'_{\alpha\beta}$ and $U'_{\alpha\beta\gamma}$, respectively. Then there is a charted 2-vector bundle $c^*\mathcal{E}$ of rank n over X with respect to $(\mathcal{U}', \mathcal{I}')$, with gluing bundles $c^*E^{\alpha\beta}$ and coherence isomorphisms $c^*\phi^{\alpha\beta\gamma}$. We say that $c^*\mathcal{E}$ is an *elementary refinement* of $\mathcal{E}$.

More generally, two charted 2-vector bundles of rank n over X are said to be *equivalent 2-vector bundles* if they can be linked by a finite chain of elementary changes of trivializations and elementary refinements. (This is the notion of equivalence that appears to be appropriate for our representability theorem 4.5.)

Remark 2.5. A charted 2-vector bundle of rank 1 consists of precisely the data defining a *gerbe* over X with band $\mathbb{C}^*$, as considered e.g. by Giraud [Gi76], Brylinski [Br93] and Hitchin [H99, §1]. There is a unitary form of the definition above, with Hermitian gluing bundles and unitary coherence isomorphisms, and a unitary 2-vector bundle of rank 1 is nothing but a gerbe with band $U(1)$. In either case, the set of equivalence classes of $\mathbb{C}^*$-gerbes or $U(1)$-gerbes over X is in natural bijection with the third integral cohomology group $H^3(X; \mathbb{Z})$ [Br93, 5.2.10].

Definition 2.6. Let $\mathcal{E} \downarrow X$ be a charted 2-vector bundle of rank n, with notation as above, and let $a\colon Y \to X$ be a map of topological spaces. Then there is a charted 2-vector bundle $a^*\mathcal{E} \downarrow Y$ of rank n obtained from $\mathcal{E}$ by *pullback* along a. It is charted with respect to the ordered open cover $(\mathcal{U}', \mathcal{I})$ with $\mathcal{U}' = \{U'_\alpha = a^{-1}(U_\alpha) \mid \alpha \in \mathcal{I}\}$. It has gluing bundles $a^*E^{\alpha\beta}$ obtained by pullback of the matrix of vector bundles $E^{\alpha\beta}$ along $a\colon U'_{\alpha\beta} \to U_{\alpha\beta}$, and coherence isomorphisms $a^*\phi^{\alpha\beta\gamma}$ obtained by pullback of the matrix of vector bundle isomorphisms $\phi^{\alpha\beta\gamma}$ along $a\colon U'_{\alpha\beta\gamma} \to U_{\alpha\beta\gamma}$. By definition there is then a *map* of charted 2-vector bundles $\hat{a}\colon a^*\mathcal{E} \to \mathcal{E}$ covering $a\colon Y \to X$.

Definition 2.7. Let $\mathcal{E} \downarrow X$ and $\mathcal{F} \downarrow X$ be charted 2-vector bundles with respect to the same ordered open cover $(\mathcal{U}, \mathcal{I})$ of X, of ranks n and m, with gluing bundles $E^{\alpha\beta}$ and $F^{\alpha\beta}$ and coherence isomorphisms $\phi^{\alpha\beta\gamma}$ and $\psi^{\alpha\beta\gamma}$, respectively. Their *Whitney sum* $\mathcal{E} \oplus \mathcal{F} \downarrow X$ is then the charted 2-vector bundle of rank $(n+m)$ with gluing bundles given by the $(n+m) \times (n+m)$ matrix of vector bundles

$$\begin{pmatrix} E^{\alpha\beta} & 0 \\ 0 & F^{\alpha\beta} \end{pmatrix}$$

and coherence isomorphisms given by the $(n+m) \times (n+m)$ matrix of vector bundle isomorphisms

$$\begin{pmatrix} \phi^{\alpha\beta\gamma} & 0 \\ 0 & \psi^{\alpha\beta\gamma} \end{pmatrix} : \begin{pmatrix} E^{\alpha\beta} & 0 \\ 0 & F^{\alpha\beta} \end{pmatrix} \cdot \begin{pmatrix} E^{\beta\gamma} & 0 \\ 0 & F^{\beta\gamma} \end{pmatrix} \xrightarrow{\cong} \begin{pmatrix} E^{\alpha\gamma} & 0 \\ 0 & F^{\alpha\gamma} \end{pmatrix} .$$

There is an elementary change of trivializations from $\mathcal{E} \oplus \mathcal{F}$ to $\mathcal{F} \oplus \mathcal{E}$ given by the $(n+m) \times (n+m)$ matrix

$$T^\alpha = \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix}$$

for each α in $\mathcal{I}$, and identity isomorphisms $\tau^{\alpha\beta}$. Here I_n denotes the identity $n \times n$ matrix, with the trivial rank 1 vector bundle in each diagonal entry and zero bundles elsewhere.

3. Algebraic K-theory of two-vector spaces

Let $(\mathcal{B}, \oplus, \otimes, \underline{0}, \underline{1})$ be a *symmetric bimonoidal category*, with sum and tensor functors

$$\oplus, \otimes \colon \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B} ,$$

and zero and unit objects $\underline{0}, \underline{1}$ in $\mathcal{B}$. These satisfy associative, commutative and distributive laws, etc., up to a list of natural isomorphisms, and these isomorphisms are *coherent* in the sense that they fulfill a (long) list of compatibility conditions, as presented by Laplaza in [L72, §1]. We say that $\mathcal{B}$ is a *bipermutative category* if the natural isomorphisms are almost all identity morphisms, except for the commutative laws for $\oplus$ and $\otimes$ and the left distributive law, and these in turn fulfill the (shorter) list of compatibility conditions listed by May in [Ma77, §VI.3].

Suppose that $\mathcal{B}$ is *small*, i.e., that the class of objects of $\mathcal{B}$ is in fact a set. Let $\pi_0(\mathcal{B})$ be the set of path components of the geometric realization of $\mathcal{B}$. (Two objects of $\mathcal{B}$ are in the same path component if and only if they can be linked by a finite chain of morphisms in $\mathcal{B}$.) Then the sum and tensor functors induce sum and product pairings that make $\pi_0(\mathcal{B})$ into a commutative semi-ring with zero and unit. We can therefore think of the symmetric bimonoidal category $\mathcal{B}$ as a kind of generalized commutative semi-ring. Conversely, any commutative semi-ring may be viewed as a discrete category, with only identity morphisms, which is then a symmetric bimonoidal category.

The additive Grothendieck group completion $\mathrm{Gr}(\pi_0(\mathcal{B}))$ of the commutative semi-ring $\pi_0(\mathcal{B})$ is a commutative ring. Likewise, the geometric realization $|\mathcal{B}|$ can be group completed with respect to the symmetric monoidal pairing induced by the sum functor $\oplus$, and this group completion can take place at the categorical level, say by Quillen's construction $\mathcal{B}^{-1}\mathcal{B}$ [Gr76] or its generalization $\mathcal{B}^+ = E\mathcal{B} \times_{\mathcal{B}} \mathcal{B}^2$ due to Thomason [T79, 4.3.1]. However, the tensor functor $\otimes$ does not readily extend to $\mathcal{B}^{-1}\mathcal{B}$, as was pointed out by Thomason [T80]. So $\mathcal{B}^{-1}\mathcal{B}$ is a symmetric monoidal category, but usually not a symmetric bimonoidal category.

Example 3.1. Let $\mathcal{V}$ be the topological bipermutative category of finite dimensional complex vector spaces, with set of objects $\mathbb{N} = \{0, 1, 2, \dots\}$ with $d \in \mathbb{N}$ interpreted as the complex vector space $\mathbb{C}^d$, and morphism spaces

$$\mathcal{V}(d, e) = \begin{cases} U(d) & \text{if } d = e, \\ \emptyset & \text{otherwise} \end{cases}$$

from d to e. The sum functor $\oplus$ takes (d, e) to $d + e$ and embeds $U(d) \times U(e)$ into $U(d + e)$ by the block sum of matrices. The tensor functor $\otimes$ takes (d, e) to de and maps $U(d) \times U(e)$ to $U(de)$ by means of the left lexicographic ordering, which identifies $\{1, \dots, d\} \times \{1, \dots, e\}$ with $\{1, \dots, de\}$. Both of these functors are continuous. The zero and unit objects are 0 and 1, respectively.

In this case, the semi-ring $\pi_0(\mathcal{V}) = \mathbb{N}$ is that of the non-negative integers, with additive group completion $\mathrm{Gr}(\mathbb{N}) = \mathbb{Z}$. The geometric realization $|\mathcal{V}| = \coprod_{d \geq 0} BU(d)$ is the classifying space for complex vector bundles, while its group completion $|\mathcal{V}^{-1}\mathcal{V}| \simeq \mathbb{Z} \times BU$ classifies virtual vector bundles. The latter space is the infinite loop space underlying the spectrum $ku = \mathrm{Spt}(\mathcal{V})$ that represents connective complex topological K-theory, which is associated to either of the symmetric monoidal categories $\mathcal{V}$ or $\mathcal{V}^{-1}\mathcal{V}$ by the procedure of Segal [Se74], as generalized by Shimada and Shimakawa [SS79] and Thomason [T79, 4.2.1].

Definition 3.2. Let $(\mathcal{B}, \oplus, \otimes, \underline{0}, \underline{1})$ be a symmetric bimonoidal category. The category $M_n(\mathcal{B})$ of *$n \times n$ matrices over* $\mathcal{B}$ has objects the matrices $V = (V_{ij})_{i,j=1}^n$ with entries that are objects of $\mathcal{B}$, and morphisms the matrices $\phi = (\phi_{ij})_{i,j=1}^n$ with entries that are morphisms in $\mathcal{B}$. The source (domain) of ϕ is the matrix of sources of the entries ϕ_{ij}, and similarly for targets (codomains).

There is a *matrix multiplication functor*

$$M_n(\mathcal{B}) \times M_n(\mathcal{B}) \xrightarrow{\cdot} M_n(\mathcal{B})$$

that takes two matrices $U = (U_{ij})_{i,j=1}^n$ and $V = (V_{jk})_{j,k=1}^n$ to the matrix $W = U \cdot V = (W_{ik})_{i,k=1}^n$ with

$$W_{ik} = \bigoplus_{j=1}^n U_{ij} \otimes V_{jk}$$

for $i, k = 1, \dots, n$. In general, we need to make a definite choice of how the n-fold sum is to be evaluated, say by bracketing from the left. When the direct sum functor is strictly associative, as in the bipermutative case, the choice does not matter.

The *unit object* I_n of $M_n(\mathcal{B})$ is the $n \times n$ matrix with unit entries $\underline{1}$ on the diagonal and zero entries $\underline{0}$ everywhere else.

Proposition 3.3. *$(M_n(\mathcal{B}), \cdot, I_n)$ is a monoidal category.*

In other words, the functor $\cdot$ is associative up to a natural associativity isomorphism

$$\underline{\alpha} \colon U \cdot (V \cdot W) \xrightarrow{\cong} (U \cdot V) \cdot W$$

and unital with respect to I_n up to natural left and right unitality isomorphisms. These are coherent, in the sense that they fulfill a list of compatibility conditions, including the Mac Lane–Stasheff pentagon axiom. The proof of the proposition is a direct application of Laplaza's first coherence theorem from [L72, §7].

Definition 3.4. Let B be a commutative semi-ring with additive Grothendieck group completion the commutative ring $A = \mathrm{Gr}(B)$. Let $M_n(A)$ and $M_n(B)$ be the multiplicative monoids of $n \times n$ matrices with entries in A and B, respectively, and let $GL_n(A) \subset M_n(A)$ be the subgroup of invertible $n \times n$ matrices with entries in A, i.e., those whose determinant is a unit in A. Let the submonoid $GL_n(B) \subset M_n(B)$ be the pullback in the diagram

$$\begin{tikzcd}
GL_n(B) \arrow[r] \arrow[d, tail] & GL_n(A) \arrow[d, tail] \\
M_n(B) \arrow[r] & M_n(A)\,.
\end{tikzcd}$$

Example 3.5. When $B = \mathbb{N}$ and $A = \mathbb{Z}$, $GL_n(\mathbb{N}) = M_n(\mathbb{N}) \cap GL_n(\mathbb{Z})$ is the monoid of $n \times n$ matrices with non-negative integer entries that are invertible as integer matrices, i.e., have determinant ± 1. It contains the elementary matrices that have entries 1 on the diagonal and in one other place, and 0 entries elsewhere. This is a larger monoid than the subgroup of units in $M_n(\mathbb{N})$, which only consists of the permutation matrices.

Definition 3.6. Let $\mathcal{B}$ be a symmetric bimonoidal category. Let $GL_n(\mathcal{B}) \subset M_n(\mathcal{B})$ be the full subcategory with objects the matrices $V = (V_{ij})_{i,j=1}^n$ whose matrix of path components $[V] = ([V_{ij}])_{i,j=1}^n$ lies in the submonoid $GL_n(\pi_0(\mathcal{B})) \subset M_n(\pi_0(\mathcal{B}))$. We call $GL_n(\mathcal{B})$ the category of *weakly invertible* $n \times n$ *matrices over* $\mathcal{B}$.

Corollary 3.7. $(GL_n(\mathcal{B}), \cdot, I_n)$ *is a monoidal category.*

Definition 3.8. Let $(\mathcal{M}, \cdot, e)$ be a monoidal category, and write $[p] = \{0 < 1 < \cdots < p\}$. The *bar construction* $B\mathcal{M}$ is a simplicial category $[p] \mapsto B_p\mathcal{M}$. In simplicial degree p the category $B_p\mathcal{M}$ has objects consisting of

(1) triangular arrays of objects $M^{\alpha\beta}$ of $\mathcal{M}$, for all $\alpha < \beta$ in $[p]$, and
(2) isomorphisms
$$\mu^{\alpha\beta\gamma} \colon M^{\alpha\beta} \cdot M^{\beta\gamma} \xrightarrow{\cong} M^{\alpha\gamma}$$
in $\mathcal{M}$, for all $\alpha < \beta < \gamma$ in $[p]$, such that
(3) the diagram of isomorphisms

$$\begin{tikzcd}
M^{\alpha\beta} \cdot (M^{\beta\gamma} \cdot M^{\gamma\delta}) \arrow[rr, "\underline{\alpha}"] \arrow[d, "id \cdot \mu^{\beta\gamma\delta}"'] & & (M^{\alpha\beta} \cdot M^{\beta\gamma}) \cdot M^{\gamma\delta} \arrow[d, "\mu^{\alpha\beta\gamma} \cdot id"] \\
M^{\alpha\beta} \cdot M^{\beta\delta} \arrow[r, "\mu^{\alpha\beta\delta}"'] & M^{\alpha\delta} & M^{\alpha\gamma} \cdot M^{\gamma\delta} \arrow[l, "\mu^{\alpha\gamma\delta}"]
\end{tikzcd}$$

commutes, for all $\alpha < \beta < \gamma < \delta$ in $[p]$.

Here $\underline{\alpha}$ is the associativity isomorphism for the monoidal operation $\cdot$ in $\mathcal{M}$.

The morphisms in $B_p\mathcal{M}$ from one object $(M_0^{\alpha\beta}, \mu_0^{\alpha\beta\gamma})$ to another $(M_1^{\alpha\beta}, \mu_1^{\alpha\beta\gamma})$ consist of a triangular array of morphisms $\phi^{\alpha\beta} \colon M_0^{\alpha\beta} \to M_1^{\alpha\beta}$ in $\mathcal{M}$ for all $\alpha < \beta$

in $[p]$, such that the diagram

$$\begin{tikzcd}
M_0^{\alpha\beta} \cdot M_0^{\beta\gamma} \arrow[r, "\mu_0^{\alpha\beta\gamma}"] \arrow[d, "\phi^{\alpha\beta}\cdot\phi^{\beta\gamma}"'] & M_0^{\alpha\gamma} \arrow[d, "\phi^{\alpha\gamma}"] \\
M_1^{\alpha\beta} \cdot M_1^{\beta\gamma} \arrow[r, "\mu_1^{\alpha\beta\gamma}"'] & M_1^{\alpha\gamma}
\end{tikzcd}$$

commutes, for all $\alpha < \beta < \gamma$ in $[p]$.

To allow for degeneracy operators f in the following paragraph, let $M^{\alpha\alpha} = e$ be the unit object of $\mathcal{M}$, let $\mu^{\alpha\alpha\beta}$ and $\mu^{\alpha\beta\beta}$ be the left and right unitality isomorphisms for $\cdot$, respectively, and let $\phi^{\alpha\alpha}$ be the identity morphism on e.

The simplicial structure on $B\mathcal{M}$ is given as follows. For each order-preserving function $f\colon [q] \to [p]$ let the functor $f^*\colon B_p\mathcal{M} \to B_q\mathcal{M}$ take the object $(M^{\alpha\beta}, \mu^{\alpha\beta\gamma})$ of $B_p\mathcal{M}$ to the object of $B_q\mathcal{M}$ that consists of the triangular array of objects $M^{f(\alpha)f(\beta)}$ for $\alpha < \beta$ in $[q]$ and the isomorphisms $\mu^{f(\alpha)f(\beta)f(\gamma)}$ for $\alpha < \beta < \gamma$ in $[q]$.

Each monoidal category $\mathcal{M}$ can be rigidified to an equivalent *strict monoidal category* $\mathcal{M}_s$, i.e., one for which the associativity isomorphism and the left and right unitality isomorphisms are all identity morphisms [ML71, XI.3.1]. The usual strict bar construction for $\mathcal{M}_s$ is a simplicial category $[p] \mapsto \mathcal{M}_s^p$, and corresponds in simplicial degree p to the full subcategory of $B_p\mathcal{M}_s$ where all the isomorphisms $\mu^{\alpha\beta\gamma}$ are identity morphisms.

Proposition 3.9. *The bar construction $B\mathcal{M}$ is equivalent to the strict bar construction $[p] \mapsto \mathcal{M}_s^p$ for any strictly monoidal rigidification $\mathcal{M}_s$ of $\mathcal{M}$.*

This justifies calling $B\mathcal{M}$ the bar construction. The proof is an application of Quillen's theorem A and the coherence theory for monoidal categories.

Definition 3.10. Let $\mathrm{Ar}\,\mathcal{M} = \mathrm{Fun}([1], \mathcal{M})$ be the *arrow category* of $\mathcal{M}$, with the morphisms of $\mathcal{M}$ as objects and commutative square diagrams in $\mathcal{M}$ as morphisms. There are obvious *source* and *target functors* $s, t\colon \mathrm{Ar}\,\mathcal{M} \to \mathcal{M}$. Let $\mathrm{Iso}\,\mathcal{M} \subset \mathrm{Ar}\,\mathcal{M}$ be the full subcategory with objects the isomorphisms of $\mathcal{M}$.

Lemma 3.11. *Let $(\mathcal{M}, \cdot, e)$ be a monoidal category. The category $B_2\mathcal{M}$ is the limit of the diagram*

$$\mathcal{M} \times \mathcal{M} \xrightarrow{\cdot} \mathcal{M} \xleftarrow{s} \mathrm{Iso}\,\mathcal{M} \xrightarrow{t} \mathcal{M}\,.$$

For $p \geq 2$ each object or morphism of $B_p\mathcal{M}$ is uniquely determined by the collection of its 2-faces in $B_2\mathcal{M}$, which is indexed by the set of monomorphisms $f\colon [2] \to [p]$.

Consider the symmetric bimonoidal category $\mathcal{B}$ as a kind of generalized semi-ring. The sum and tensor operations in $\mathcal{B}$ make the product category $\mathcal{B}^n$ a generalized (right) module over $\mathcal{B}$, for each non-negative integer n. The collection of $\mathcal{B}$-module homomorphisms $\mathcal{B}^n \to \mathcal{B}^n$ is encoded in terms of (left) matrix multiplication by the monoidal category $M_n(\mathcal{B})$, and we shall interpret the monoidal subcategory $GL_n(\mathcal{B})$ as a category of *weak equivalences* $\mathcal{B}^n \xrightarrow{\sim} \mathcal{B}^n$. This motivates the following definition.

Definition 3.12. Let $\mathcal{B}$ be a symmetric bimonoidal category. The *algebraic K-theory* of the 2-category of (finitely generated free) modules over $\mathcal{B}$ is the loop space

$$K(\mathcal{B}) = \Omega B\big(\coprod_{n\geq 0} |BGL_n(\mathcal{B})|\big)\,.$$

Here $|BGL_n(\mathcal{B})|$ is the geometric realization of the bar construction on the monoidal category $GL_n(\mathcal{B})$ of weakly invertible $n \times n$ matrices over $\mathcal{B}$. The block sum of matrices $GL_n(\mathcal{B}) \times GL_m(\mathcal{B}) \to GL_{n+m}(\mathcal{B})$ makes the coproduct $\coprod_{n\geq 0} |BGL_n(\mathcal{B})|$ a topological monoid. The looped bar construction ΩB provides a group completion of this topological monoid.

When $\mathcal{B} = \mathcal{V}$ is the category of finite dimensional complex vector spaces, the (finitely generated free) modules over $\mathcal{V}$ are called *2-vector spaces*, and $K(\mathcal{V})$ is the algebraic K-theory of the 2-category of 2-vector spaces.

Let $GL_\infty(\mathcal{B}) = \operatorname{colim}_n GL_n(\mathcal{B})$ be the infinite stabilization with respect to block sum with the unit object in $GL_1(\mathcal{B})$, and write $B = \pi_0(\mathcal{B})$ and $A = \operatorname{Gr}(B)$. Then $K(\mathcal{B}) \simeq \mathbb{Z} \times |BGL_\infty(\mathcal{B})|^+$ by the McDuff–Segal group completion theorem [MS76]. Here the superscript '+' refers to Quillen's plus-construction with respect to the (maximal perfect) commutator subgroup of $GL_\infty(A) \cong \pi_1|BGL_\infty(\mathcal{B})|$; cf. proposition 5.3 below.

4. Represented two-vector bundles

Let X be a topological space, with an ordered open cover $(\mathcal{U}, \mathcal{I})$. Recall that all morphisms in $\mathcal{V}$ are isomorphisms, so $\operatorname{Ar} GL_n(\mathcal{V}) = \operatorname{Iso} GL_n(\mathcal{V})$.

Definition 4.1. A *represented 2-vector bundle* $\mathcal{E}$ of rank n over X consists of

(1) a *gluing map*

$$g^{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow |GL_n(\mathcal{V})|$$

for each pair $\alpha < \beta$ in $\mathcal{I}$, and

(2) a *coherence map*

$$h^{\alpha\beta\gamma} \colon U_{\alpha\beta\gamma} \longrightarrow |\operatorname{Ar} GL_n(\mathcal{V})|$$

satisfying $s \circ h^{\alpha\beta\gamma} = g^{\alpha\beta} \cdot g^{\beta\gamma}$ and $t \circ h^{\alpha\beta\gamma} = g^{\alpha\gamma}$ over $U_{\alpha\beta\gamma}$, for each triple $\alpha < \beta < \gamma$ in $\mathcal{I}$, such that

(3) the 2-*cocycle condition*

$$h^{\alpha\gamma\delta} \circ (h^{\alpha\beta\gamma} \cdot id) \circ \underline{\alpha} = h^{\alpha\beta\delta} \circ (id \cdot h^{\beta\gamma\delta})$$

holds over $U_{\alpha\beta\gamma\delta}$ for all $\alpha < \beta < \gamma < \delta$ in $\mathcal{I}$.

There is a suitably defined notion of *equivalence* of represented 2-vector bundles, which we omit to formulate here, but cf. definitions 2.3 and 2.4.

Definition 4.2. Let $E(d) = EU(d) \times_{U(d)} \mathbb{C}^d \downarrow BU(d)$ be the universal $\mathbb{C}^d$-bundle over $BU(d)$. There is a universal $n \times n$ matrix

$$E \quad (E)^n$$

of Hermitian vector bundles over $|GL_n(\mathcal{V})|$. Over the path component $|GL_n(\mathcal{V})_D| = \prod_{i,j=1}^n BU(d_{ij})$ for $D = (d_{ij})_{i,j=1}^n$ in $GL_n(\mathbb{N})$, the (i,j)-th entry in E is the pullback of the universal bundle $E(d_{ij})$ along the projection $|GL_n(\mathcal{V})_D| \to BU(d_{ij})$.

Let $|\mathrm{Ar}\,U(d)|$ be the geometric realization of the arrow category $\mathrm{Ar}\,U(d)$, where $U(d)$ is viewed as a topological groupoid with one object. Each pair $(A,B) \in U(d)^2$ defines a morphism from C to $(A,B)\cdot C = BCA^{-1}$, so

$$|\mathrm{Ar}\,U(d)| \cong EU(d)^2 \times_{U(d)^2} U(d)$$

equals the Borel construction for this (left) action of $U(d)^2$ on $U(d)$. There are source and target maps $s, t\colon |\mathrm{Ar}\,U(d)| \to BU(d)$, which take the 1-simplex represented by a morphism (A,B) to the 1-simplices represented by the morphisms A and B, respectively. By considering each element in $U(d)$ as a unitary isomorphism $\mathbb{C}^d \to \mathbb{C}^d$ one obtains a universal unitary vector bundle isomorphism $\phi(d)\colon s^*E(d) \xrightarrow{\cong} t^*E(d)$

There is a universal $n \times n$ matrix of unitary vector bundle isomorphisms

$$\phi\colon s^*E \cong t^*E$$

over $|\mathrm{Ar}\,GL_n(\mathcal{V})|$. Over the path component $|\mathrm{Ar}\,GL_n(\mathcal{V})_D| = \prod_{i,j=1}^n |\mathrm{Ar}\,U(d_{ij})|$ for D as above, the (i,j)-th entry in ϕ is the pullback of the universal isomorphism $\phi(d_{ij})$ along the projection $|\mathrm{Ar}\,GL_n(\mathcal{V})_D| \to |\mathrm{Ar}\,U(d_{ij})|$.

Lemma 4.3. *Let $\mathcal{E}$ be a represented 2-vector bundle with gluing maps $g^{\alpha\beta}$ and coherence maps $h^{\alpha\beta\gamma}$. There is an associated charted 2-vector bundle with gluing bundles*

$$E^{\alpha\beta} = (g^{\alpha\beta})^*(E)$$

over $U_{\alpha\beta}$ and coherence isomorphisms

$$\phi^{\alpha\beta\gamma} = (h^{\alpha\beta\gamma})^*(\phi)\colon E^{\alpha\beta}\cdot E^{\beta\gamma} = (g^{\alpha\beta}\cdot g^{\beta\gamma})^*(E) \xrightarrow{\cong} (g^{\alpha\gamma})^*(E) = E^{\alpha\gamma}$$

over $U_{\alpha\beta\gamma}$. The association induces a bijection between the equivalence classes of represented 2-vector bundles and the equivalence classes of charted 2-vector bundles of rank n over X.

Definition 4.4. Let $2\text{-Vect}_n(X)$ be the set of equivalence classes of 2-vector bundles of rank n over X. For path-connected X let $2\text{-Vect}(X) = \coprod_{n\ge 0} 2\text{-Vect}_n(X)$. Whitney sum (definition 2.7) defines a pairing that makes $2\text{-Vect}(X)$ an abelian monoid.

Theorem 4.5. *Let X be a finite CW complex. There are natural bijections*

$$2\text{-Vect}_n(X) \cong [X, |BGL_n(\mathcal{V})|]$$

and

$$2\text{-Vect}(X) \cong [X, \coprod_{n\ge 0} |BGL_n(\mathcal{V})|]\,.$$

To explain the first correspondence, from which the second follows, we use the following construction.

Definition 4.6. Let $(\mathcal{U}, \mathcal{J})$ be an ordered open cover of X. The *Mayer–Vietoris blow-up* $MV(\mathcal{U})$ of X with respect to $\mathcal{U}$ is the simplicial space with p-simplices

$$MV_p(\mathcal{U}) = \coprod_{\alpha_0 \le \dots \le \alpha_p} U_{\alpha_0 \dots \alpha_p}$$

with $\alpha_0 \le \dots \le \alpha_p$ in $\mathcal{J}$. The i-th simplicial face map is a coproduct of inclusions $U_{\alpha_0 \dots \alpha_p} \subset U_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_p}$, and similarly for the degeneracy maps. The inclusions $U_{\alpha_0 \dots \alpha_p} \subset X$ combine to a natural map $e\colon |MV(\mathcal{U})| \to X$, which is a (weak) homotopy equivalence.

Sketch of proof of theorem 4.5. By lemma 3.11, a simplicial map $g\colon MV(\mathcal{U}) \to |BGL_n(\mathcal{V})|$ is uniquely determined by its components in simplicial degrees 1 and 2. The first of these is a map

$$g_1\colon MV_1(\mathcal{U}) = \coprod_{\alpha \le \beta} U_{\alpha\beta} \to |B_1 GL_n(\mathcal{V})| = |GL_n(\mathcal{V})|$$

which is a coproduct of gluing maps $g^{\alpha\beta}\colon U_{\alpha\beta} \to |GL_n(\mathcal{V})|$. The second is a map

$$g_2\colon MV_2(\mathcal{U}) = \coprod_{\alpha \le \beta \le \gamma} U_{\alpha\beta\gamma} \to |B_2 GL_n(\mathcal{V})| \, .$$

The simplicial identities and lemma 3.11 imply that g_2 is determined by g_1 and a coproduct of coherence maps $h^{\alpha\beta\gamma}\colon U_{\alpha\beta\gamma} \to |\mathrm{Ar}\, GL_n(\mathcal{V})|$. Hence such a simplicial map g corresponds bijectively to a represented 2-vector bundle of rank n over X.

Any map $f\colon X \to |BGL_n(\mathcal{V})|$ can be composed with the weak equivalence $e\colon |MV(\mathcal{U})| \to X$ to give a map of spaces $fe\colon |MV(\mathcal{U})| \to |BGL_n(\mathcal{V})|$, which is homotopic to a simplicial map g if $\mathcal{U}$ is a good cover, and for reasonable X any open ordered cover can be refined to a good one. The homotopy class of f corresponds to the equivalence class of the represented 2-vector bundle determined by the simplicial map g. $\square$

Remark 4.7. We wish to interpret the 2-vector bundles over X as (effective) 0-cycles for some cohomology theory at X. Such theories are group-valued, so a first approximation to the 0-th cohomology group at X could be the Grothendieck group $\mathrm{Gr}(2\text{-}\mathrm{Vect}(X))$ of formal differences of 2-vector bundles over X. The analogous construction for ordinary vector bundles works well to define topological K-theory, but for 2-vector bundles this algebraically group completed functor is not even representable, like in the case of the algebraic K-theory of a discrete ring. We thank Haynes Miller for reminding us of this issue.

Instead we follow Quillen and perform the group completion at the space level, which leads to the algebraic K-theory space

$$\begin{aligned} K(\mathcal{V}) &= \Omega B\big(\coprod_{n \ge 0} |BGL_n(\mathcal{V})|\big) \\ &\simeq \mathbb{Z} \times |BGL_\infty(\mathcal{V})|^+ \end{aligned}$$

from definition 3.12. But what theory does this loop space represent? One interpretation is provided by the theory of virtual flat fibrations, presented by Karoubi in [K87, Ch. III], leading to what we shall call virtual 2-vector bundles. Another interpretation could be given using the homology bordism theory of Hausmann and Vogel [HV78].

Definition 4.8. Let X be a space. An *acyclic fibration* over X is a Serre fibration $a\colon Y \to X$ such that the homotopy fiber at each point $x \in X$ has the integral homology of a point, i.e., $\tilde{H}_*(\mathrm{hofib}_x(a);\mathbb{Z}) = 0$. A map of acyclic fibrations from $a'\colon Y' \to X$ to $a\colon Y \to X$ is a map $f\colon Y' \to Y$ with $af = a'$.

A *virtual 2-vector bundle* over X is described by an acyclic fibration $a\colon Y \to X$ and a 2-vector bundle $\mathcal{E} \downarrow Y$. We write $\mathcal{E} \downarrow Y \xrightarrow{a} X$. Given a map $f\colon Y' \to Y$ of acyclic fibrations over X there is an induced 2-vector bundle $f^*\mathcal{E} \downarrow Y'$. The virtual 2-vector bundles described by $\mathcal{E} \downarrow Y \xrightarrow{a} X$ and $f^*\mathcal{E} \downarrow Y' \xrightarrow{a'} X$ are declared to be *equivalent* as virtual 2-vector bundles over X.

Lemma 4.9. *The abelian monoid of equivalence classes of virtual 2-vector bundles over X is the colimit*

$$\operatorname*{colim}_{a\colon Y \to X} \text{2-Vect}(Y)$$

where $a\colon Y \to X$ ranges over the category of acyclic fibrations over X. Its Grothendieck group completion is isomorphic to the colimit

$$\operatorname*{colim}_{a\colon Y \to X} \mathrm{Gr}(\text{2-Vect}(Y))\,.$$

The functor $Y \mapsto \text{2-Vect}(Y)$ factors through the homotopy category of acyclic fibrations over X, which is directed.

The following result says that formal differences of virtual 2-vector bundles over X are the geometric objects that constitute cycles for the contravariant homotopy functor represented by the algebraic K-theory space $K(\mathcal{V})$. Compare [K87, III.3.11]. So $K(\mathcal{V})$ represents sheaf cohomology for the topology of acyclic fibrations, with coefficients in the abelian presheaf $Y \mapsto \mathrm{Gr}(\text{2-Vect}(Y))$ given by the Grothendieck group completion of the abelian monoid of equivalence classes of 2-vector bundles.

Theorem 4.10. *Let X be a finite CW complex. There is a natural group isomorphism*

$$\operatorname*{colim}_{a\colon Y \to X} \mathrm{Gr}(\text{2-Vect}(Y)) \cong [X, K(\mathcal{V})]$$

where $a\colon Y \to X$ ranges over the category of acyclic fibrations over X. Restricted to $\mathrm{Gr}(\text{2-Vect}(X))$ (with $a = id$) the isomorphism extends the canonical monoid homomorphism $\text{2-Vect}(X) \cong [X, \coprod_{n\ge 0} |BGL_n(\mathcal{V})|] \to [X, K(\mathcal{V})]$.

Remark 4.11. The passage to sheaf cohomology would be unnecessary if we replaced $\mathcal{V}$ by a different symmetric bimonoidal category $\mathcal{B}$ such that each $\pi_0(GL_n(\mathcal{B}))$ is abelian. This might entail an extension of the category of vector spaces to allow generalized vector spaces of arbitrary real, or even complex, dimension, parallel to the inclusion of the integers into the real or complex numbers. Such an extension is reminiscent of a category of representations of a suitable C^*-algebra, but we know of no clear interpretation of this approach.

5. Algebraic K-theory of topological K-theory

Is the contravariant homotopy functor $X \mapsto [X, K(\mathcal{V})] = K(\mathcal{V})^0(X)$ part of a cohomology theory, and if so, what is the spectrum representing that theory?

The topological symmetric bimonoidal category $\mathcal{V}$ plays the role of a generalized

$\mathcal{V}^{-1}\mathcal{V}$ correspondingly plays the role of a generalized commutative ring. This may be tricky to realize at the level of symmetric bimonoidal categories, but the connective topological K-theory spectrum $ku = \mathrm{Spt}(\mathcal{V})$ associated to the additive topological symmetric monoidal structure of $\mathcal{V}$ is an E_∞ ring spectrum, and hence a commutative algebra over the sphere spectrum $\mathbb{S}$.

The algebraic K-theory of an $\mathbb{S}$-algebra $\mathbb{A}$ can on one hand be defined as the Waldhausen algebraic K-theory [W85] of a category with cofibrations and weak equivalences, with objects the finite cell $\mathbb{A}$-modules, morphisms the $\mathbb{A}$-module maps and weak equivalences the stable equivalences. Alternatively, it can be defined as a group completion

$$K(\mathbb{A}) = \Omega B\big(\coprod_{n\geq 0} B\widehat{GL}_n(\mathbb{A})\big)$$

where $\widehat{GL}_n(\mathbb{A})$ is essentially the topological monoid of $\mathbb{A}$-module maps $\mathbb{A}^n \to \mathbb{A}^n$ that are stable equivalences. The former definition produces a spectrum, so the space $K(\mathbb{A})$ is in fact an infinite loop space, and its deloopings represent a cohomology theory.

The passage from modules over the semi-ring object $\mathcal{V}$ to modules over the ring object ku corresponds to maps $|GL_n(\mathcal{V})| \to \widehat{GL}_n(ku)$ and a map $K(\mathcal{V}) \to K(ku)$.

Conjecture 5.1. *There is a weak equivalence $K(\mathcal{V}) \simeq K(ku)$. More generally, $K(\mathcal{B}) \simeq K(\mathbb{A})$ for each symmetric bimonoidal category $\mathcal{B}$ with associated commutative $\mathbb{S}$-algebra $\mathbb{A} = \mathrm{Spt}(\mathcal{B})$*

Remark 5.2. The conjecture asserts that the contravariant homotopy functor $X \mapsto [X, K(\mathcal{V})]$ with 0-cycles given by the virtual 2-vector bundles over X is the 0-th cohomology group for the cohomology theory represented by the spectrum $K(ku)$ given by the algebraic K-theory of connective topological K-theory. We consider the virtual 2-vector bundles over X to be sufficiently geometric objects (like complex vector bundles), that this cohomology theory then admits as geometric an interpretation as the classical examples of de Rham cohomology, topological K-theory and complex bordism.

As a first (weak) justification of this conjecture, recall that to the eyes of algebraic K-theory the block sum operation $(g, h) \mapsto \begin{bmatrix} g & 0 \\ 0 & h \end{bmatrix}$ is identified with the stabilized matrix multiplication $(g, h) \mapsto \begin{bmatrix} gh & 0 \\ 0 & I \end{bmatrix}$, where I is an identity matrix. The group completion in the definition of algebraic K-theory adjoins inverses to the block sum operation, and thus also to the stabilized matrix multiplication. In particular, for each elementary $n \times n$ matrix $e_{ij}(V)$ over $\mathcal{B}$ with (i, j)-th off-diagonal entry equal to an object V of $\mathcal{B}$, the inverse matrix $e_{ij}(-V)$ is formally adjoined as far as algebraic K-theory is concerned. Hence the formal negatives $-V$ in $\mathcal{B}^{-1}\mathcal{B}$ are already present, in this weak sense.

A stronger indication that the conjecture should hold true is provided by the following special case. Recall that a commutative semi-ring is the same as a (small) symmetric bimonoidal category that is *discrete*, i.e., has only identity morphisms.

Proposition 5.3. *Let B be a commutative semi-ring, with additive Grothendieck group completion $A = \mathrm{Gr}(B)$. The semi-ring homomorphism $B \to A$ induces a weak equivalence*

$$BGL_\infty(B) \xrightarrow{\simeq} BGL_\infty(A)$$

and thus a weak equivalence $K(B) \simeq K(A)$. In particular, there is a weak equivalence $K(\mathbb{N}) \simeq K(\mathbb{Z})$.

A proof uses the following application of Quillen's theorem B.

Lemma 5.4. *Let $f\colon M \to G$ be a monoid homomorphism from a monoid M to a group G. Write $mg = f(m) \cdot g$. Let $Q = B(*, M, G)$ be the category with objects $g \in G$ and morphisms $(m, g) \in M \times G$ from mg to g:*

$$mg \xrightarrow{(m,g)} g\,.$$

Then there is a fiber sequence up to homotopy

$$|Q| \longrightarrow BM \xrightarrow{Bf} BG\,.$$

Sketch of proof of proposition 5.3. Applying lemma 5.4 to the monoids $M_n = GL_n(B)$ and groups $G_n = GL_n(A)$ we obtain categories Q_n for each natural number n. There are stabilization maps $i\colon Q_n \to Q_{n+1}$, $M_n \to M_{n+1}$ and $G_n \to G_{n+1}$, with (homotopy) colimits Q_∞, M_∞ and G_∞, and a quasi-fibration

$$|Q_\infty| \longrightarrow BGL_\infty(B) \longrightarrow BGL_\infty(A)\,.$$

It suffices to show that each stabilization map $i\colon |Q_n| \to |Q_{n+1}|$ is weakly null-homotopic, because then $|Q_\infty|$ is weakly contractible.

For each full subcategory $K \subset Q_n$ with finitely many objects, the restricted stabilization functor $i|K$ takes g to $i(g) = \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}$. It receives a natural transformation from a functor $j\colon K \to Q_{n+1}$ that maps g to $j(g) = \begin{bmatrix} g & v \\ 0 & 1 \end{bmatrix}$ for some column vector $v = v(g)$ with positive entries in B. The trick is to construct $v(g)$ inductively for the finite set of objects g of K, so that $v(mg)$ is sufficiently positive with respect to $m \cdot v(g)$ for all morphisms $mg \to g$ in K.

Furthermore, the finiteness of K ensures that there is a row vector w with entries in B and an object $h = \begin{bmatrix} I_n & 0 \\ -w & 1 \end{bmatrix}$ of Q_{n+1} such that there is a natural transformation from j to the constant functor to h. These two natural transformations provide a homotopy from $i|K$ to a constant map. As K was arbitrary with finitely many objects, this means that i is weakly null-homotopic. $\square$

Remark 5.5. If there exists a symmetric bimonoidal category $\mathcal{W}$ and a functor $\mathcal{V} \to \mathcal{W}$ of symmetric bimonoidal categories that induces an additive equivalence from $\mathcal{V}^{-1}\mathcal{V}$ to $\mathcal{W}$, then most likely the line of argument sketched above in the case of commutative semi-rings can be adapted to the symmetric bimonoidal situation. This could provide one line of proof toward conjecture 5.1. Similar remarks apply for a general symmetric bimonoidal category $\mathcal{B}$ in place of $\mathcal{V}$.

6. Forms of elliptic cohomology

In this section we shall view the algebraic K-theory $K(\mathbb{A})$ of an $\mathbb{S}$-algebra $\mathbb{A}$ as a spectrum, rather than as a space.

We shall argue that the algebraic K-theory $K(ku)$ of the connective topological K-theory spectrum ku is a connective form of elliptic cohomology, in the sense that

it detects homotopy theoretic phenomena related to v_2-periodicity, much like how topological K-theory detects phenomena related to v_1-periodicity (which is really the same as Bott periodicity) and how rational cohomology detects phenomena related to v_0-periodicity. Furthermore, from this point of view the homotopy type of $K(ku)$ is robust with respect to changes in the interpretation of the phrase "algebraic K-theory of topological K-theory".

We first introduce a filtration of the class of spectra that is related to the *chromatic filtration* given by the property of being Bousfield local with respect to some Johnson–Wilson theory $E(n)$ (cf. Ravenel [Ra92, §7]), but is more appropriate for the connective spectra that arise from algebraic K-theory. Our notion is also more closely linked to aspects of v_n-periodicity than to being $E(n)$-local.

Let p be a prime, $K(n)$ the n-th Morava K-theory at p and F a p-local finite CW spectrum. The least number $0 \le n \le \infty$ such that $K(n)_*(F)$ is non-trivial is called the *chromatic type* of F. By the Hopkins–Smith periodicity theorem [HS98, Thm. 9], F admits a *v_n-self map* $v \colon \Sigma^d F \to F$ such that $K(m)_*(v)$ is an isomorphism for $m = n$ and zero for $m \ne n$. The v_n-self map is sufficiently unique for the mapping telescope

$$v^{-1}F = \mathrm{Tel}\big(F \xrightarrow{v} \Sigma^{-d}F \xrightarrow{v} \dots\big)$$

to be well-defined up to homotopy. The class of all p-local finite CW spectra of chromatic type $\ge n$ is closed under weak equivalences and the formation of homotopy cofibers, desuspensions and retracts, so we say that the full subcategory that it generates is a *thick* subcategory. By the Hopkins–Smith thick subcategory theorem [HS98, Thm. 7], any thick subcategory of the category of p-local finite CW spectra has this precise form, for a unique number $0 \le n \le \infty$.

Definition 6.1. Let X be a spectrum, and let $\mathcal{T}_X$ be the full subcategory of p-local finite CW spectra F for which the localization map

$$F \wedge X \longrightarrow v^{-1}F \wedge X$$

induces an isomorphism on homotopy groups in all sufficiently high degrees. Then $\mathcal{T}_X$ is a thick subcategory, hence consists of the spectra F of chromatic type $\ge n$ for some unique number $0 \le n \le \infty$. We call this number $n = \mathrm{telecom}(X)$ the *telescopic complexity* of X. (This abbreviation is due to Matthew Ando.)

Lemma 6.2. *If Y is the k-connected cover of X, for some integer k, then X and Y have the same telescopic complexity.*

Let $X \to Y \to Z$ be a cofiber sequence and $m = \max\{\mathrm{telecom}(X), \mathrm{telecom}(Y)\}$. If $\mathrm{telecom}(X) \ne \mathrm{telecom}(Y)$ then $\mathrm{telecom}(Z) = m$, otherwise $\mathrm{telecom}(Z) \le m$.

If Y is a (de-)suspension of X then X and Y have the same telescopic complexity.

If Y is a retract of X then $\mathrm{telecom}(Y) \le \mathrm{telecom}(X)$.

If X is an $E(n)$-local spectrum then X has telescopic complexity $\le n$.

Examples 6.3. (1) Integral, rational, real and complex cohomology ($H\mathbb{Z}$, $H\mathbb{Q}$, $H\mathbb{R}$ or $H\mathbb{C}$) all have telescopic complexity 0.

(2) Connective or periodic, real or complex topological K-theory (ko, ku, KO or KU) all have telescopic complexity 1. The étale K-theory $K^{et}(R)$ of a ring $R = \mathcal{O}$ of S-integers in a local or global number field has telescopic complexity 1.

and so does the algebraic K-theory $K(R)$ if the Lichtenbaum–Quillen conjecture holds for the ring R.

(3) An Ando–Hopkins–Strickland [AHS01] *elliptic spectrum* (E, C, t) has telescopic complexity ≤ 2, and the telescopic complexity equals 2 if and only if the elliptic curve C over $R = \pi_0(E)$ has a supersingular specialization over some point of $\mathrm{Spec}(R)$.

(4) The Hopkins–Mahowald–Miller *topological modular forms* spectra tmf and TMF have telescopic complexity 2.

(5) The Johnson–Wilson spectrum $E(n)$ and its connective form, the Brown–Peterson spectrum $BP\langle n\rangle$, both have telescopic complexity n.

(6) The sphere spectrum $\mathbb{S}$ and the complex bordism spectrum MU have infinite telescopic complexity.

Let $V(1)$ be the four-cell Smith–Toda spectrum with $BP_*(V(1)) = BP_*/(p, v_1)$. For $p \geq 5$ it exists as a commutative ring spectrum. It has chromatic type 2, and there is a v_2-self map $v\colon \Sigma^{2p^2-2}V(1) \to V(1)$ inducing multiplication by the class $v_2 \in \pi_{2p^2-2}V(1)$. We write $V(1)_*(X) = \pi_*(V(1) \wedge X)$ for the $V(1)$-homotopy groups of X, which are naturally a graded module over $P(v_2) = \mathbb{F}_p[v_2]$.

Let $X_{(p)}$ and $X_p^\wedge$ denote the p-localization and p-completion of a spectrum X, respectively. The first Brown–Peterson spectrum $\ell = BP\langle 1\rangle$ is the connective p-local Adams summand of $ku_{(p)}$, and its p-completion $\ell_p^\wedge$ is the connective p-complete Adams summand of $ku_p^\wedge$. These are all known to be commutative $\mathbb{S}$-algebras.

The spectrum $TC(\ell_p^\wedge)$ occurring in the following statement is the *topological cyclic homology* of $\ell_p^\wedge$, as defined by Bökstedt, Hsiang and Madsen [BHM93]. The theorem is proved in [AR02, 0.3] by an elaborate but explicit computation of its $V(1)$-homotopy groups, starting from the corresponding $V(1)$-homotopy groups of the topological Hochschild homology $THH(\ell_p^\wedge)$.

Theorem 6.4 (Ausoni–Rognes). *Let $p \geq 5$. The algebraic K-theory spectrum $K(\ell_p^\wedge)$ of the connective p-complete Adams summand $\ell_p^\wedge$ has telescopic complexity* 2. *More precisely, there is an exact sequence of $P(v_2)$-modules*

$$0 \to \Sigma^{2p-3}\mathbb{F}_p \to V(1)_*K(\ell_p^\wedge) \xrightarrow{\mathrm{trc}} V(1)_*TC(\ell_p^\wedge) \to \Sigma^{-1}\mathbb{F}_p \to 0$$

and an isomorphism of $P(v_2)$-modules

$$\begin{aligned} V(1)_*TC(\ell_p^\wedge) \cong P(v_2) \otimes \\ \big(E(\partial, \lambda_1, \lambda_2) \oplus E(\lambda_2)\{\lambda_1 t^d \mid 0 < d < p\} \oplus E(\lambda_1)\{\lambda_2 t^{dp} \mid 0 < d < p\}\big)\,. \end{aligned}$$

Here ∂, t, λ_1 and λ_2 have degrees -1, -2, $2p-1$ and $2p^2-1$, respectively. Hence $V(1)_*TC(\ell_p^\wedge)$ is free of rank $(4p+4)$ over $P(v_2)$, and agrees with its v_2-localization in sufficiently high degrees.

Since $K(\ell_p^\wedge)$ has telescopic complexity 2, it has a chance to detect v_2-periodic families in $\pi_*V(1)$. This is indeed the case. Let $\alpha_1 \in \pi_{2p-3}V(1)$ and $\beta'_1 \in \pi_{2p^2-2p-1}V(1)$ be the classes represented in the Adams spectral sequence by the cobar 1-cycles $h_{10} = [\bar\xi_1]$ and $h_{11} = [\bar\xi_1^p]$, respectively. There are maps $V(1) \to v_2^{-1}V(1) \to L_2V(1)$, and Ravenel [Ra86, 6.3.22] computed

$$\pi_* L_2V(1) \cong P(v_2, v_2^{-1}) \otimes E(\zeta)\{1, h_{10}, h_{11}, g_0, g_1, h_{10}g_0, h_{11}g_1\}$$

for $p \geq 5$. Hence $\pi_* L_2 V(1)$ contains twelve v_2-periodic families. The telescope conjecture asserted that $v_2^{-1}V(1) \to L_2V(1)$ might be an equivalence, but this is now considered to be unlikely [MRS01]. The following detection result can be read off from [AR02, 4.8], and shows that $K(\ell_p^\wedge)$ detects the same kind of homotopy theoretic phenomena as $E(2)$ or an elliptic spectrum.

Proposition 6.5. *The unit map $\mathbb{S} \to K(\ell_p^\wedge)$ induces a $P(v_2)$-module homomorphism $\pi_*V(1) \to V(1)_*K(\ell_p^\wedge)$ which takes 1, α_1 and β_1' to 1, $t\lambda_1$ and $t^p\lambda_2$, respectively. Hence $V(1)_*K(\ell_p^\wedge)$ detects the v_2-periodic families in $\pi_*V(1)$ generated by these three classes.*

Turning to the whole connective p-complete topological K-theory spectrum $ku_p^\wedge$, there is a map $\ell_p^\wedge \to ku_p^\wedge$ of commutative $\mathbb{S}$-algebras. It induces a natural map $K(\ell_p^\wedge) \to K(ku_p^\wedge)$, and there is a transfer map $K(ku_p^\wedge) \to K(\ell_p^\wedge)$ such that the composite self-map of $K(\ell_p^\wedge)$ is multiplication by $(p-1)$. Hence the composite map is a p-local equivalence.

Lemma 6.6. *The algebraic K-theory spectrum $K(ku_p^\wedge)$ of connective p-complete topological K-theory $ku_p^\wedge$ contains $K(\ell_p^\wedge)$ as a p-local retract, hence has telescopic complexity ≥ 2.*

Most likely $K(ku_p^\wedge)$ also has telescopic complexity exactly 2. It may be possible to prove this directly by computing $V(1)_*TC(ku_p^\wedge)$, by similar methods as in [AR02], but the algebra involved for $ku_p^\wedge$ is much more intricate than it was for the Adams summand. Some progress in this direction has recently been made by Ausoni.

The following consequence of a theorem of the second author [D97, p. 224] allows us to compare the algebraic K-theory of $ku_p^\wedge$ to that of the integral spectra ku and $K(\mathbb{C})$.

Theorem 6.7 (Dundas). *Let $\mathbb{A}$ be a connective $\mathbb{S}$-algebra. The commutative square*

$$\begin{tikzcd}
K(\mathbb{A}) \arrow[r] \arrow[d] & K(\mathbb{A}_p^\wedge) \arrow[d] \\
K(\pi_0(\mathbb{A})) \arrow[r] & K(\pi_0(\mathbb{A}_p^\wedge))
\end{tikzcd}$$

becomes homotopy Cartesian after p-completion.

We apply this with $\mathbb{A} = ku$ or $\mathbb{A} = K(\mathbb{C})$. Also in the second case $\mathbb{A}_p^\wedge \simeq ku_p^\wedge$, by Suslin's theorem on the algebraic K-theory of algebraically closed fields [Su83]. Then $\pi_0(\mathbb{A}) = \mathbb{Z}$ and $\pi_0(\mathbb{A}_p^\wedge) = \mathbb{Z}_p$. It is known that $K(\mathbb{Z}_p)$ has telescopic complexity 1, by Bökstedt–Madsen [BM94], [BM95] for p odd and by the third author [Ro99] for $p = 2$. It is also known that $K(\mathbb{Z})$ has telescopic complexity 1 for $p = 2$, by Voevodsky's proof of the Milnor conjecture and Rognes–Weibel [RW00]. For p odd it would follow from the Lichtenbaum–Quillen conjecture for $K(\mathbb{Z})$ at p that $K(\mathbb{Z})$ has telescopic complexity 1, and this now seems to be close to a theorem by the work of Voevodsky, Rost and Positselski.

Proposition 6.8. *Suppose that $K(\mathbb{Z})$ has telescopic complexity 1 at a prime $p \geq 5$. Then $K(ku)$ and $K(K(\mathbb{C}))$ have the same telescopic complexity as $K(ku_p^\wedge)$, which is ≥ 2.*

More generally it is natural to expect that $K(K(R))$ has telescopic complexity 2 for each ring of S-integers $R = \mathcal{O}_{F,S}$ in a local or global number field F, including the initial case $K(K(\mathbb{Z}))$. A discussion of such a conjecture has been given in lectures by the third author, but should take place in the context of étale covers or Galois extensions of commutative $\mathbb{S}$-algebras, which would take us too far afield here.

The difference between the connective and periodic topological K-theory spectra ku and KU may also not affect their algebraic K-theories greatly. There is a (localization) fiber sequence

$$K(\mathcal{C}^{KU}(ku)) \to K(ku) \to K(KU)$$

where $\mathcal{C}^{KU}(ku)$ is the category of finite cell ku-module spectra that become contractible when induced up to KU-modules. Such spectra have finite Postnikov towers with layers that are induced from finite cell $H\mathbb{Z}$-module spectra via the map $ku \to H\mathbb{Z}$, and so it is reasonable to expect that a generalized form of the devissage theorem in algebraic K-theory applies to identify $K(\mathcal{C}^{KU}(ku))$ with $K(\mathbb{Z})$.

Proposition 6.9. *If there is a fiber sequence $K(\mathbb{Z}) \to K(ku) \to K(KU)$ and $K(\mathbb{Z})$ has telescopic complexity* 1*, at a prime $p \geq 5$, then the algebraic K-theory spectrum $K(KU)$ of the periodic topological K-theory spectrum KU has the same telescopic complexity as $K(ku)$, which is ≥ 2.*

Remark 6.10. Unlike traditional elliptic cohomology, the spectrum $K(ku)$ is not complex orientable. For the map of $\mathbb{S}$-algebras $ku \to H\mathbb{Z}$ is 2-connected, hence induces a 3-connected map $K(ku) \to K(\mathbb{Z})$; cf. [W78, 1.1], [BM94, 10.9]. So the composite map $\mathbb{S} \to K(ku) \to K(\mathbb{Z})$ detects $\eta \in \pi_1(\mathbb{S})$. This implies that the unit map for $K(ku)$ cannot factor through the complex bordism spectrum MU, since $\pi_1(MU) = 0$. This should not be perceived as a problem, however, as e.g. also the topological modular forms spectrum tmf is not complex orientable. It seems more likely that $K(\overline{KU})$ can be complex oriented, where $\overline{KU}$ is an "algebraic closure" of KU in the category of commutative $\mathbb{S}$-algebras.

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